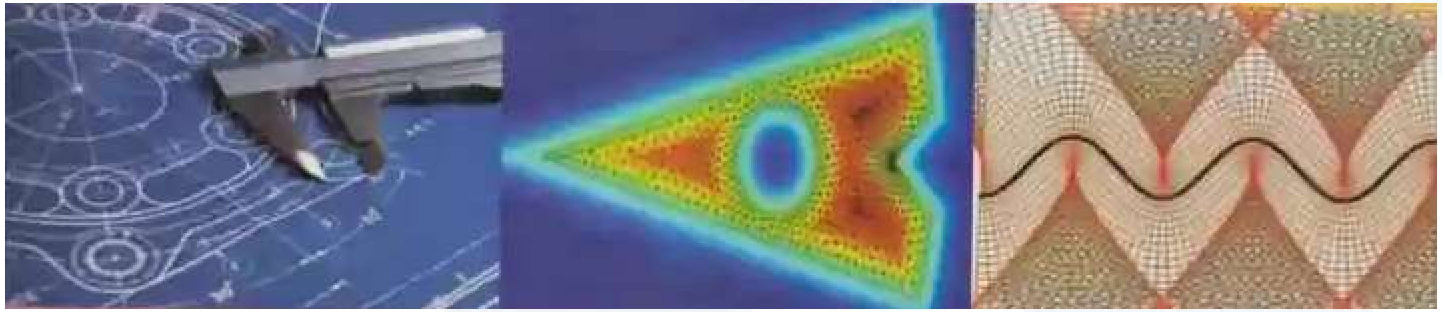
CEDC301: Mathematics Engineering

Exercises 1 & 2: Functions of a Complex Variable



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1. If $f(z) = z^{24} - 3z^{20} + 4z^{12} - 5z^{6}$, find $f\left(\dfrac{1+i}{\sqrt{2}}\right)$

2. Let z and w be complex numbers such that $|z| = 1$ and $|w| \neq 1$. Prove that

$$\left|\frac{z-\omega}{1-z\bar{\omega}}\right| = 1$$

3. Using the epsilon-delta definition, Prove that $\lim\limits_{z \to 1+i} [(1-i)z + 2i] = 2 + 2i$

4. Show that the given limit does not exist

$$\lim_{z \to 0}\left(\frac{2xy}{x^2+y^2} - \frac{y^2}{x^2} i\right)$$

5. Show that the given limit does not exist

$$\lim_{z\to 0}\left(\frac{z}{\bar{z}}\right)^2$$

6. Determine whether or not the limit exists

$$\lim_{z\to 0}\left(\frac{2y^2}{x^2}-\frac{x^2-y^2}{y^2}i\right)$$

7. Show that the function $f(z) = -(2xy + 5x) + i(x^2 - 5y - y^2)$ is analytic for all z. Find $f'(z)$.

8. Determine whether the function $f(z) = x^3 + xy^2 - 4x + i(4y - y^3 - x^2y)$ is differentiable. Is it analytic?

1. Find all values of z satisfying the given equation

$e^{z-1} = -ie^2$ $\qquad$ $e^{2z} + e^z + 1 = 0$

2. Find all values of the given quantity

$(1+i)^{(1+i)}$ $\qquad$ $(1+\sqrt{3}i)^{3i}$

3. Find all values of z satisfying the given equation

$\sin z = 2$ $\qquad$ $\cos z = -3i$ $\qquad$ $\sinh z = -i$ $\qquad$ $\cos z = i \sin z$

4. Verify that $(i^i)^2 = i^{2i}$, but $(i^2)^i \neq i^{2i}$

5. Find where $\tan^{-1} z = \frac{i}{2} \log \frac{i+z}{i-z}$ is analytic?

6. Find the derivative of the principal value of the given function at the given point.

$$z^{3/2};\ z = 1 + i \qquad\qquad z^{1+i};\ z = 1 + \sqrt{3}i$$

7. Find all values of the given quantity

$$\sinh^{-1} i \qquad\qquad \tanh^{-1}(i + 1)$$

1. If $f(z) = z^{24} - 3z^{20} + 4z^{12} - 5z^{6}$, find $f\left(\frac{1+i}{\sqrt{2}}\right)$

$$\frac{1+i}{\sqrt{2}} = e^{i\pi/4}$$

$$z^{24} = e^{i6\pi} = 1, \quad z^{20} = e^{i5\pi} = -1, \quad z^{12} = e^{i3\pi} = -1, \quad z^{6} = e^{i3\pi/2} = -i$$

$$f\left(\frac{1+i}{\sqrt{2}}\right) = 1 - 3(-1) + 4(-1) - 5(-i) = 5i$$

2. Let z and w be complex numbers such that $|z| = 1$ and $|w| \neq 1$. Prove that

$$\left|\frac{z-\omega}{1-z\overline{\omega}}\right| = 1$$

$$\left|\frac{z-\omega}{1-z\overline{\omega}}\right|^2 = \frac{z-\omega}{1-z\overline{\omega}}\frac{\overline{z}-\overline{\omega}}{1-\overline{z}\omega} = \frac{z\overline{z}-z\overline{\omega}-\omega\overline{z}+\omega\overline{\omega}}{1-\overline{z}\omega-z\overline{\omega}+z\overline{z}\omega\overline{\omega}} = \frac{1-z\overline{\omega}-\omega\overline{z}+|\omega|^2}{1-\overline{z}\omega-z\overline{\omega}+|\omega|^2} = 1$$

3. Using the epsilon-delta definition, Prove that

$$\lim_{z \to 1+i} [(1-i)z + 2i] = 2 + 2i$$

$\forall\ \varepsilon > 0$, $\exists ?\ \delta > 0$ such that $|[(1-i)z + 2i] - (2+2i)| < \varepsilon$ whenever $0 < |z - (1+i)| < \delta$

$$\left|[(1-i)z + 2i] - (2+2i)\right| < \varepsilon \Rightarrow \left|(1-i)z - 2\right| < \varepsilon$$

$$|1-i|\left|z - \frac{2}{1-i}\right| < \varepsilon \Rightarrow \sqrt{2}\left|z - (1+i)\right| < \varepsilon$$

$$\left|z - (1+i)\right| < \frac{\varepsilon}{\sqrt{2}} = \delta$$

4. Show that the given limit does not exist

$$\lim_{z\to 0}\left(\frac{2xy}{x^2+y^2}-\frac{y^2}{x^2}i\right)$$

z approach 0 along the line $y=x$: $\lim_{z\to 0}\left(\frac{2xy}{x^2+y^2}-\frac{y^2}{x^2}i\right)=\lim_{x\to 0}\left(\frac{2x^2}{2x^2}-\frac{x^2}{x^2}i\right)=1-i$

z approach 0 along the line $y=2x$:

$$\lim_{z\to 0}\left(\frac{2xy}{x^2+y^2}-\frac{y^2}{x^2}i\right)=\lim_{x\to 0}\left(\frac{4x^2}{5x^2}-\frac{4x^2}{x^2}i\right)=\frac{4}{5}-4i$$

5. Show that the given limit does not exist

$$\lim_{z\to 0}\left(\frac{z}{\bar{z}}\right)^2$$

z approach 0 along the real axis: $\lim_{z\to 0}\left(\frac{z}{\bar{z}}\right)^2 = \lim_{x\to 0}\left(\frac{x}{x}\right)^2 = 1$

z approach 0 along the imaginary axis: $\lim_{z\to 0}\left(\frac{z}{\bar{z}}\right)^2 = \lim_{y\to 0}\left(\frac{iy}{-iy}\right)^2 = 1$

z approach 0 along the line $y = x$: $\lim_{z\to 0}\left(\frac{z}{\bar{z}}\right)^2 = \lim_{x\to 0}\left(\frac{x+ix}{x-ix}\right)^2 = -1$

6. Determine whether or not the limit exists $\lim_{z\to 0}\left(\frac{2y^2}{x^2}-\frac{x^2-y^2}{y^2}i\right)$

z approach 0 along the line $y=x$:

$$\lim_{z\to 0}\left(\frac{2y^2}{x^2}-\frac{x^2-y^2}{y^2}i\right)=\lim_{x\to 0}\left(\frac{2x^2}{x^2}-\frac{x^2-x^2}{x^2}i\right)=2$$

z approach 0 along the line $y=-x$:

$$\lim_{z\to 0}\left(\frac{2y^2}{x^2}-\frac{x^2-y^2}{y^2}i\right)=\lim_{x\to 0}\left(\frac{2x^2}{x^2}-\frac{x^2-x^2}{x^2}i\right)=2$$

z approach 0 along the line $y=2x$:

$$\lim_{z\to 0}\left(\frac{2y^2}{x^2}-\frac{x^2-y^2}{y^2}i\right)=\lim_{x\to 0}\left(\frac{8x^2}{x^2}-\frac{x^2-4x^2}{x^2}i\right)=8+3i$$

7. Show that the function $f(z) = -(2xy + 5x) + i(x^2 - 5y - y^2)$ is analytic for all z. Find $f'(z)$.

$$\frac{\partial u}{\partial x} = -2y - 5 = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -2x = -\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = (-2y - 5) + i(2x)$$

8. Determine whether the function $f(z) = x^3 + xy^2 - 4x + i(4y - y^3 - x^2y)$ is differentiable. Is it analytic?

$$\frac{\partial u}{\partial x} = 3x^2 + y^2 - 4, \quad \frac{\partial v}{\partial y} = 4 - 3y^2 - x^2$$

$$\frac{\partial u}{\partial y} = 2xy, \quad \frac{\partial v}{\partial x} = -2xy \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 3x^2 + y^2 - 4 = 4 - 3y^2 - x^2 \Rightarrow x^2 + y^2 = 2$$

Continuity of u, v, and the first partial derivatives guarantee f is differentiable on the circle. However, f is nowhere analytic.

1. Find all values of z satisfying the given equation

$$e^{z-1} = -ie^2 \qquad\qquad e^{2z} + e^z + 1 = 0$$

$$z-1 = \log(-ie^2) = \ln(e^2) + i\left(\frac{3\pi}{2} + 2n\pi\right) = 2 + i\left(\frac{3\pi}{2} + 2n\pi\right)$$

$$z = 3 + i\left(\frac{3\pi}{2} + 2n\pi\right)$$

$$e^z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$z = \log\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \left(\frac{2\pi}{3} + 2n\pi\right)i, \qquad z = \log\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \left(\frac{4\pi}{3} + 2n\pi\right)i$$

2. Find all values of the given quantity

$(1+i)^{(1+i)}$ $\qquad$ $(1+\sqrt{3}i)^{3i}$

$$(1+i)^{(1+i)} = e^{(1+i)\log(1+i)} = e^{(1+i)[\ln\sqrt{2}+i(\pi/4+2n\pi)]}$$

$$= e^{\ln\sqrt{2}-(\pi/4+2n\pi)}\left[\cos(\pi/4+\ln\sqrt{2})+i\sin(\pi/4+\ln\sqrt{2})\right] = e^{-2n\pi}[0.274+0.584i]$$

$$(1+\sqrt{3}i)^{3i} = e^{3i\log(1+\sqrt{3}i)} = e^{3i[\ln 2+i(\pi/3+2n\pi)]}$$

$$= e^{3\ln 2i-3(\pi/3+2n\pi)]} = e^{-(\pi+6n\pi)}\left[\cos(3\ln 2)+i\sin(3\ln 2)\right] = e^{-6n\pi}[-0.021+0.038i]$$

3. Find all values of z satisfying the given equation

$\sin z = 2$ $\cos z = -3i$ $\sinh z = -i$ $\cos z = i \sin z$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2} = 2 \Rightarrow e^{2iz} - 4ie^{iz} - 1 = 0 \Rightarrow e^{iz} = 2i \pm \sqrt{3}i$$

$$iz = \ln[(2 \pm \sqrt{3})i] \Rightarrow z = -i[\ln(2 \pm \sqrt{3}) + (\pi/2 + 2n\pi)i]$$

$$z = \pi/2 + 2n\pi - i\ln(2 \pm \sqrt{3})$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = -3i \Rightarrow e^{2iz} + 6ie^{iz} + 1 = 0 \Rightarrow e^{iz} = -3i \pm \sqrt{10}i$$

$$iz = \ln[(-3 \pm \sqrt{10})i] \Rightarrow z = -i[\ln(\sqrt{10} - 3) + (\pi/2 + 2n\pi)i]$$

$$z = \pi/2 + 2n\pi - i\ln(\sqrt{10} - 3)$$

$$\sinh z = \frac{e^{z} - e^{-z}}{2} = -i \Rightarrow e^{2iz} + 2ie^{iz} - 1 = 0 \Rightarrow e^{iz} = -i$$

$$z = \ln(i) = -(\pi/2 + 2n\pi)i$$

$$z = \pi/2 + 2n\pi - i\ln(2 \pm \sqrt{3})$$

$$\cos z = i\sin z \Rightarrow e^{iz} + e^{-iz} = e^{iz} - e^{-iz} \Rightarrow e^{-iz} = 0$$

Since this last equation has no solutions, the original equation has no solutions.

4. Verify that $(i^i)^2 = i^{2i}$, but $(i^2)^i \neq i^{2i}$

$$(i^i)^2 = (e^{i\ln i})^2 = [e^{-(\pi/2+2\pi n)}]^2 = e^{-(\pi+4\pi n)}, \quad i^{2i} = e^{2i\ln i} = e^{-(\pi+4\pi n)}$$

$$(i^2)^i = (-1)^i = e^{i\ln(-1)} = e^{-(\pi+2\pi n)}, \quad i^{2i} = e^{2i\ln i} = e^{-(\pi+4\pi n)}$$

5. Find where $\tan^{-1} z = \frac{i}{2}\log\frac{i+z}{i-z}$ is analytic?

$$\left\{z: \frac{i+z}{i-z} = w \in (-\infty, 0)\right\} = \left\{z: z = i\frac{w-1}{w+1}, w \in (-\infty, 0)\right\}$$

For $w \in (-\infty, 0]$,

$$\frac{w-1}{w+1} = 1 - \frac{2}{w+1} \in (-\infty, -1] \cup (1, \infty)$$

So $\tan^{-1} z$ is analytic in

$$\mathbb{C} \setminus \{z: Re(z) = 0, Im(z) \in (-\infty, -1] \cup (1, \infty)\}$$

6. Find the derivative of the principal value of the given function at the given point.

$z^{3/2}$; $z = 1 + i$

$$\frac{3}{2}\sqrt[4]{2}e^{i\pi/8}$$

z^{1+i}; $z = 1 + \sqrt{3}i$

$$\sqrt{2}e^{-\pi/3+i[\pi/4+\ln 2]}$$

7. Find all values of the given quantity

$\sinh^{-1} i$

$$\frac{1}{2}(4n+1)\pi i,\ n \in \mathbb{Z}$$

$\tanh^{-1}(i+1)$

$$\frac{1}{4}\ln 2 + \frac{1}{8}(8n+3)\pi i,\ n \in \mathbb{Z}$$